REMARKS ON MIRROR SYMMETRY OF DONALDSON-THOMAS THEORY FOR CALABI-YAU 4-FOLDS

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Abstract. Motivated by Strominger-Yau-Zaslow's mirror symmetry proposal and Kontsevich's homological mirror symmetry conjecture, we study mirror phenomena (in A-model) of certain results from Donaldson-Thomas theory for Calabi-Yau 4-folds.

1. Introduction

Mirror symmetry is a duality between symplectic geometry (A-model) and complex geometry (B-model) for Calabi-Yau manifolds [52]. In the B-model, Donaldson-Thomas invariants [17, 47, 25, 31] count holomorphic bundles (or coherent sheaves) on Calabi-Yau 3-folds. Borisov and Joyce [6] and the authors [9, 10, 11, 12] studied their extensions to Calabi-Yau 4-folds (abbrev. CY_4).

The purpose of this note is to study certain corresponding mirror phenomena in the A-model for CY_4, mainly motivated by Strominger-Yau-Zaslow's geometric mirror symmetry proposal [46, 36], Kontsevich's homological mirror symmetry conjecture [28] and Thomas' paper on CY_3 [48]. In particular, we study calibrated geometry [21] for CY_4 and point out corresponding structures in DT_4 theory (B-model). We continue Thomas' table [48] as follows.

Topological twists	B-model	A-model
Calabi-Yau 4-folds	$\check{X}$	X
Complex/ Symplectic structures	$\Omega=\Omega_{\check{X}}\in H^{4,0}(\check{X})$ $\omega=\omega_{\check{X}}\in H^{1,1}(\check{X})$	$\omega=\omega_X\in H^{1,1}(X)$ $\Omega=\Omega_X\in H^{4,0}(X)$
Geometric objects	Connections on a vector bundle $E\rightarrow\check{X}$	Submanifolds in class $[L]\in H^4(X)$ with connections on $E\rightarrow L$
Star operators	Choose a metric h_E on E $*_4\triangleq(\Omega\lrcorner)\circ *_{h_E}\circlearrowleft\Omega^{0,\bullet}(\check{X},EndE)$	Choose a metric h on E, $g_L=g_X\|_L$ $*_{g_L}\circlearrowleft\Omega^{\bullet}(L),\quad *_h\circlearrowleft\Omega^{\bullet}(L,\mathfrak{g}_E)$
Energy functionals	$\int_{\check{X}}\|F^{0,2}\|^2_{h_E}dvol$	$\int_L(\|F\|^2_h+\|\omega\|_L\|^2)dvol_{g_L}$
Energy minimizers	$F^{0,2}+*_4F^{0,2}=0$ Complex ASD connections	$\omega\|_L+*_{g_L}(\omega\|_L)=0,\ \ F^+=0$ ASD submfds with ASD bundles
Reductions	If $ch_2(E)\in\mathrm{Ker}(\wedge[\Omega])\cap H^4(\check{X})$ $F^{0,2}_+=0\Rightarrow F^{0,2}=0$	If $[L]\in\mathrm{Ker}(\wedge[\omega^2])\cap H^4(X)$ $(\omega\|_L)^+=0\Rightarrow\omega\|_L=0$
Moment maps	$F\wedge\omega^3$	$\mathrm{Im}(\Omega)\|_L$

The ASD submanifolds mentioned in the above table (see section 2) are corresponding mirror objects of complex ASD connections on CY_4. To continue the discussion, let us first fix the following notation.

Notation 1.1. Unless specified otherwise, we denote
(1) X to be a Calabi-Yau 4-fold (compact or convex at infinity with $c_1(X)=0$);
(2) L to be a compact relatively spin Lagrangian submanifold in X with zero Maslov index.

In the definition of DT_4 invariants (B-model), Brav-Bussi-Joyce's local Darboux theorem [7] (see Theorem 4.5) for moduli spaces of simple sheaves on CY_4 is an important ingredient, which

says for any simple sheaf $\mathcal{F}$, we could choose a local Kuranishi map

$$\kappa : Ext^1(\mathcal{F},\mathcal{F}) \to Ext^2(\mathcal{F},\mathcal{F})$$

such that

$$\int_X Tr(\kappa \cup \kappa) \cup \Omega_X = 0.$$

We are interested in the corresponding mirror result in the A-model. In fact, the analog of the above Kuranishi map in A-model is

$$\kappa : H^1(L;\Lambda^+_{0,nov}) \to H^2(L;\Lambda^+_{0,nov}),$$

$$\kappa(x) \triangleq \sum_{k=0}^{\infty} m_k(x^{\otimes k}),$$

where $\{m_k\}_{k\geq 0}$ is the A_∞-algebra structure on $H^*(L;\Lambda^+_{0,nov})$ defined by Fukaya [18].

Theorem 1.2. *(Theorem 3.1)*
Let $L \subseteq X$ be a Lagrangian submanifold in a CY_4. Then

$$Q(\kappa,\kappa) = const,$$

where Q is the Poincaré pairing on $H^2(L;\Lambda^+_{0,nov})$.

This theorem follows from a combination of a general result for cyclic A_∞-algebras (see Lemma 3.2) and the existence of a cyclic A_∞-structure on $H^*(L;\Lambda^+_{0,nov})$ [18].

If L is an *unobstructed* Lagrangian, i.e. there exists $b \in H^1(L;\Lambda^+_{0,nov})$ such that $\kappa(b) = 0$, one can define the twisted A_∞-algebra $(H^*(L;\Lambda^+_{0,nov}), m^b_k)$ with

$$m^b_k(x_1,\cdots,x_k) = \sum_{n\geq k}\sum m_n(b,\cdots,b,x_1,b,\cdots,b,\cdots,x_k,b,\cdots,b),$$

where the first summation is taken over all such expressions. The corresponding Kuranishi map

$$\kappa^b : H^1(L;\Lambda^+_{0,nov}) \to H^2(L;\Lambda^+_{0,nov}), \quad \kappa^b(x) = \sum_{k=0}^{\infty} m^b_k(x^{\otimes k})$$

is similarly defined.

$(H^*(L;\Lambda^+_{0,nov}), m^b_k)$ is a cyclic A_∞-algebra with $m^b_0(1) = 0$ provided $(H^*(L;\Lambda^+_{0,nov}), m_k)$ is a cyclic A_∞-algebra (see also [18]). As a corollary of the above theorem, we get an unobstructedness result for moduli spaces of Maurer-Cartan elements, i.e. if the space of bounding cochains b's is nonempty, then it is the whole $H^1(L;\Lambda^+_{0,nov})$.

Theorem 1.3. *(Theorem 3.3)*
Let $L \subseteq X$ be a definite [1] and unobstructed Lagrangian submanifold in a CY_4. Then
(1) $\kappa \equiv 0$; (2) for any $b \in H^1(L;\Lambda^+_{0,nov})$, $\kappa^b \equiv 0$.

The outline of this note is as follows: In section 2, we introduce the Harvey-Lawson (anti)-self-dual submanifolds in CY_4 and study their basic properties. We also discuss an orientability problem for moduli spaces of special Lagrangian submanifolds. In section 3, we study FOOO's Lagrangian Floer theory on CY_4 and point out corresponding structures in the B-side. In the final section, we recall basic facts in DT_4 theory (B-side story).

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[1] i.e. the intersection form on $H^2(L,\mathbb{R})$ is definite.

2. Mirror aspects of DT_4 theory in calibrated geometry

2.1. **Harvey-Lawson (anti-)self-duals in eight manifolds.** We recall that the mirror of holomorphic bundles (resp. HYM bundles) on Calabi-Yau manifolds are Lagrangian submanifolds (resp. special Lagrangian submanifolds) coupled with flat bundles. In complex 4-dimension, under SYZ mirror symmetry [46] [36], solutions to the DT_4 equations

$$\begin{cases} F_+^{0,2} = 0 \\ F \wedge \omega^3 = 0, \end{cases}$$

become special Harvey-Lawson ASD submanifolds coupled with ASD bundles as described below.

Definition 2.1. Given an almost Hermitian eight manifold[2] (X, g, J, ω), an oriented four-dimensional submanifold L is a Harvey-Lawson anti-self-dual submanifold if

$$(\omega|_L)^+ \triangleq \frac{1}{2}(\omega|_L + *(\omega|_L)) = 0 \in \Omega_+^2(L),$$

where $*$ is the Hodge-star operator on L for the induced metric $g|_L$.

When (X, g, J, ω) is a CY_4 with holomorphic volume form Ω, a Harvey-Lawson ASD submanifold L is special if it satisfies

$$Im(\Omega)|_L = 0.$$

Remark 2.2. This notion was introduced by Harvey-Lawson [21] for submanifolds in $\mathbb{C}^4$. They also showed special ASD submanifolds are exactly the same as Cayley submanifolds with respect to the Cayley 4-form $Re(\Omega) - \frac{1}{2}\omega^2$.

If $d\omega = 0$, i.e. X is almost Kähler, Lagrangian submanifolds are Harvey-Lawson ASD's. A converse statement is given by

Proposition 2.3. *Let (X, g, J, ω) be an almost Kähler eight manifold, L be a closed Harvey-Lawson ASD submanifold such that $[(\omega|_L)^2] = 0 \in H^4(L)$. Then L is a Lagrangian submanifold.*

Proof. By [16], we have an identity

$$(\omega|_L)^2 = (|(\omega|_L)^+|^2 - |(\omega|_L)^-|^2) dvol_L.$$

From the Stokes theorem and $[(\omega|_L)^2] = 0 \in H^4(L)$, we obtain

$$0 = \int_L (\omega|_L)^2 = -\int_L |(\omega|_L)^-|^2 dvol_L.$$

Finally, we use the energy identity

$$\int_L |(\omega|_L)|^2 = \int_L (|(\omega|_L)^+|^2 + |(\omega|_L)^-|^2) dvol_L$$

to get the conclusion. □

Remark 2.4. Any Harvey-Lawson ASD with $b_2 = 0$ is a Lagrangian submanifold.

Remark 2.5. Besides Lagrangian submanifolds, half-dimensional almost Kähler submanifolds L's are also examples of Harvey-Lawson self-dual's, because $\omega|_L$ is the almost Kähler form of L which is a self-dual two form on $(L, g|_L)$ [16]. This shows Harvey-Lawson ASD's could have obstructed deformations in general.

Remark 2.6. (Harvey-Lawson ASD's under geometric flows)
Lagrangian submanifolds in Kähler-Einstein manifolds (e.g. Calabi-Yau manifolds) are preserved under the mean curvature flow whose stationary points are minimal Lagrangians (they are special Lagrangians in the Calabi-Yau case). For general Kähler manifolds, one need to couple the Kähler-Ricci flow with the mean curvature flow to preserve the Lagrangian condition [44].

Lotay and Pacini [37] extended the above result to totally real submanifolds in almost Kähler manifolds by coupling the symplectic curvature flow [45] (a generalization of Kähler-Ricci flow) with the Maslov flow (a generalization of MCF).

In a Kähler-Einstein manifold, Maslov flow preserves the pull-back of the Kähler form to any totally real submanifold [3]. If we use a fixed metric instead of the induced metric, the flow preserves totally real Harvey-Lawson ASD's.

[2] It is an almost complex manifold with a Hermitian metric.

[3] Totally real is an open condition in the Grassmannian of all 4-planes inside $\mathbb{C}^4$.

2.2. **Orientations for moduli spaces of special Lagrangians in Calabi-Yau manifolds.** In this section, we study the mirror of the orientability result for moduli spaces of sheaves on Calabi-Yau manifolds [11]. We first recall the moment map approach to the moduli space of (special-)Lagrangians in Calabi-Yau manifolds, which is the beautiful work of Donaldson [15] and Hitchin [23].

Let L be a closed n-manifold with a fixed volume form $dvol_L$, and X be a Calabi-Yau n-fold with a Kähler form ω and a holomorphic volume form Ω. We consider the space $\mathrm{Map}_0(L,X)$ of smooth maps f's with $f^*[\omega]=0\in H^2(L)$, and a symplectic form φ on it defined by

$$\varphi|_{(f)}:\Omega^0(L,f^*TX)\otimes\Omega^0(L,f^*TX)\rightarrow\mathbb{R},$$

$$\varphi|_{(f)}(v_1,v_2)=\int_L\omega(v_1,v_2)dvol_L.$$

The group $\mathcal{G}=\mathrm{Diff}_{dvol_L}(L)$ of volume-preserving diffeomorphisms acts on $\mathrm{Map}_0(L,X)$ preserving the symplectic form φ. The zero loci of the corresponding moment map consists precisely of those maps f's with $f^*(\omega)=0$.

The complex structure on X induces a complex structure on $\mathrm{Map}_0(L,X)$, and the subspace

$$S=\{f\in\mathrm{Map}_0(L,X)\mid f^*(\Omega)=dvol_L\}$$

is a complex submanifold of $\mathrm{Map}_0(L,X)$ consisting of immersions. We take the subgroup $\mathcal{G}_0\subseteq\mathcal{G}$ to be the kernel of the Calabi map [15], [4]. The symplectic quotient $\mathcal{M}^c\triangleq S//\mathcal{G}_0$ is a Lagrangian torus bundle (with fiber $H_1(L,\mathbb{R})/H_1(L,\mathbb{Z})$) over the moduli space $\mathcal{M}$ of (immersed) special Lagrangian submanifolds. $\mathcal{M}$ has an integral affine structure by the Arnold-Liouville theorem.

We define vector bundles $E^*=(S\times H^*(L,\mathbb{C}))//\mathcal{G}_0$ over $\mathcal{M}^c$, and form the determinant complex line bundle $\mathcal{L}=det(E^*)\rightarrow\mathcal{M}^c$.

Proposition 2.7.
(1) if n is even, $c_1(\mathcal{L})=0$ provided that $H_1(\mathcal{M}^c,\mathbb{Z})$ has no 2-torsion elements,
(2) if n is odd, $\mathcal{L}$ has a square root.

Proof. (1) As $\mathcal{G}_0$ preserves the volume form on L, the Poincaré pairing on $H^*(L,\mathbb{C})$ induces an isomorphism $\mathcal{L}\cong\mathcal{L}^*$ between complex line bundles when n is even. Since $H^2(\mathcal{M}^c,\mathbb{Z})$ has no 2-torsion elements, $2c_1(\mathcal{L})=0\Rightarrow c_1(\mathcal{L})=0$.

(2) If n is odd, $det(E^{odd})$ gives a square root of $\mathcal{L}$ by Poincaré duality. □

3. Mirror aspects of DT_4 theory in Lagrangian Floer theory

3.1. **Mirror results.** In this section, we study mirror phenomena of DT_4 theory from the perspective of Lagrangian Floer theory. Lagrangian Floer cohomology $HF^*(L)$, introduced by Fukaya, Oh, Ohta and Ono [20], is defined in terms of counting holomorphic disks bounding Lagrangian submanifold L. Given a Calabi-Yau mirror pair $(X,\check{X})$, there should exist a correspondence

$$Ext^*_{\check{X}}(\mathcal{F},\mathcal{F})\leftrightarrow HF^*(L)$$

under mirror symmetry [28]. In good cases, $HF^*(L)\cong H^*(L)$ and Serre duality pairing in the B-model would be mirror to the Poincaré pairing in the A-model. On CY_3, moduli spaces of $\mathcal{F}$ (resp. (L,b)[4]) are locally critical points of holomorphic functions [7], [25] (resp. [19]). On CY_4, we have local 'Darboux models' for moduli spaces of simple sheaves (Theorem 4.5), we expect a similar structure in the A-model.

To state the result, we first introduce the Kuranishi map

$$\kappa:H^1(L;\Lambda^+_{0,nov})\rightarrow H^2(L;\Lambda^+_{0,nov}),$$

$$\kappa(x)\triangleq\sum_{k=0}^{\infty}m_k(x^{\otimes k}),$$

where $\{m_k\}_{k\geq0}$ is the A_∞-algebra structure on $H^*(L;\Lambda^+_{0,nov})$ defined by Fukaya [18].

Theorem 3.1. *Let $L\subseteq X$ be a Lagrangian submanifold in a CY_4. Then*

$$Q(\kappa,\kappa)=const,$$

where Q is the Poincaré pairing on $H^2(L;\Lambda^+_{0,nov})$.

In fact, this result follows from a combination of the existence of a cyclic A_∞-structure on $H^*(L;\Lambda^+_{0,nov})$ due to Fukaya [18] (see Theorem 3.7) and the following lemma on cyclic A_∞-algebras.

[4] b is a bounding cochain which helps to define $HF^*(L)$ (see Fukaya [19]).

Lemma 3.2. *Given a cyclic A_∞-algebra (A, m_k, Q), for any $k \geq 0$ and $x \in A^1$, we have*
$$\sum_{k_1+k_2=k+1} Q(m_{k_1}(x^{\otimes k_1}), m_{k_2}(x^{\otimes k_2})) = 0.$$
In particular, $Q(\kappa, \kappa) = Q(m_0(1), m_0(1))$, where $\kappa : A^1 \to A^2$, $\kappa(x) \triangleq \sum_{k=0}^{\infty} m_k(x^{\otimes k})$ is the Kuranishi map of (A, m_k).

Proof. From Definition 3.6, given $k_1, k_2 \geq 0$ with $k_1 + k_2 \geq 1$, we have
$$Q(m_{k_1}(x^{\otimes k_1}), m_{k_2}(x^{\otimes k_2})) = Q(m_{k_1}(x^{\otimes r}, m_{k_2}(x^{\otimes k_2}), x^{\otimes t}), x),$$
for $r, t \geq 0$ with $r + t + 1 = k_1$. We fix $k_1 + k_2 = k + 1 \geq 1$, then
$$(\frac{k+1}{2}) \sum_{k_1+k_2=k+1} Q(m_{k_1}(x^{\otimes k_1}), m_{k_2}(x^{\otimes k_2})) = \sum_{k_1+k_2=k+1} \sum_{r+t+1=k_1} Q(m_{k_1}(x^{\otimes r}, m_{k_2}(x^{\otimes k_2}), x^{\otimes t}), x),$$
which is zero by the A_∞-relation. □

On CY_4, local 'Darboux models' for moduli spaces of stable sheaves (Theorem 4.5) have an application to the unobstructedness of these moduli spaces (Corollary 4.6). We expect a similar result for moduli spaces of Maurer-Cartan elements of A_∞-algebras $H^*(L; \Lambda^+_{0,nov})$'s (one could work with non-archemedian geometry to make sense the moduli space as done in [18]).

By SYZ mirror symmetry proposal [46], [36] and Kontsevich's HMS conjecture [28], a sheaf (with Ext^* group) in the B-model is mirror to a Lagrangian (we take the flat bundle to be trivial for simplicity) with a bounding cochain (i.e. a Maurer-Cartan element which helps to define HF^*) in the A-model. As a corollary of Theorem 3.1, the unobstructedness result in the A-model should be stated as follows.

We start with an *unobstructed* Lagrangian [5], define the twisted A_∞-algebra $(H^*(L; \Lambda^+_{0,nov}), m^b_k)$
$$m^b_k(x_1, \cdots, x_k) = \sum_{n \geq k} \sum m_n(b, \cdots, b, x_1, b, \cdots, b, \cdots, x_k, b, \cdots, b),$$
where the first summation is taken over all such expressions. The corresponding Kuranishi map
$$\kappa^b : H^1(L; \Lambda^+_{0,nov}) \to H^2(L; \Lambda^+_{0,nov}), \quad \kappa^b(x) = \sum_{k=0}^{\infty} m^b_k(x^{\otimes k})$$
is similarly defined.

$(H^*(L; \Lambda^+_{0,nov}), m^b_k)$ is a cyclic A_∞-algebra with $m^b_0(1) = 0$ provided that $(H^*(L; \Lambda^+_{0,nov}), m_k)$ is a cyclic A_∞-algebra [18]. The unobstructedness result says if the space of bounding cochains b's is nonempty, then it is the whole $H^1(L; \Lambda^+_{0,nov})$, i.e.

Theorem 3.3. *Let $L \subseteq X$ be a definite[6] and unobstructed Lagrangian in a CY_4. Then (1) $\kappa \equiv 0$; (2) for any $b \in H^1(L; \Lambda^+_{0,nov})$, $\kappa^b \equiv 0$.*

Proof. (1) Since L is unobstructed, there exists b with $\kappa(b) = \sum_{k=0}^{\infty} m_k(b^{\otimes k}) = 0$. By Theorem 3.1, we have
$$Q(\kappa, \kappa) = Q(m_0(1), m_0(1)) = Q(\kappa(b), \kappa(b)) = 0.$$
The definite quadratic form Q on $H^2(L, \mathbb{R})$ gives $\kappa \equiv 0$, i.e. any element $b \in H^1(L; \Lambda^+_{0,nov})$ is a bounding cochain.

(2) We define the twisted A_∞-algebra $(H^*(L; \Lambda^+_{0,nov}), m^b_k)$ which is still cyclic [18]. As $m^b_0(1) = \kappa(b) = 0$, we apply Lemma 3.2 to $(H^*(L; \Lambda^+_{0,nov}), m^b_k)$ and obtain $Q(\kappa^b, \kappa^b) = 0$. By the definite quadratic form on $H^2(L, \mathbb{R})$, we have $\kappa^b \equiv 0$. □

Remark 3.4. By Donaldson's renowned theorem [13], [14], definite intersection forms on closed smooth 4-manifolds are diagonalizable over integers.

On some particular type of CY_4, say K_Y, where Y is a compact Fano 3-fold, local Kuranishi maps for deformations of (compactly supported) stable sheaves have more refined structures than local 'Darboux models' in Theorem 4.5 (see Lemma 6.4 [10]). The refined structure is deduced from the cyclic completion structure on $Ext^*(\iota_*\mathcal{F}, \iota_*\mathcal{F})$ [40]. In general, on the canonical bundle K_Y of a compact Fano n-fold Y, for any coherent sheaf $\mathcal{F}$, we have
$$Ext^*_{K_Y}(\iota_*\mathcal{F}, \iota_*\mathcal{F}) \cong Ext^*_Y(\mathcal{F}, \mathcal{F}) \oplus Ext^{n+1-*}_Y(\mathcal{F}, \mathcal{F})^*. \tag{1}$$

[5] i.e. there exists $b \in H^1(L; \Lambda^+_{0,nov})$ such that $\kappa(b) = 0$.

[6] i.e. the intersection form on $H^2(L, \mathbb{R})$ is definite.

We are interested in its mirror analog in Lagrangian Floer theory (A-model), and take $Y = \mathbb{P}^n$ as an example, whose mirror is given by a superpotential [29], [24]

$$W = \sum_{i=1}^{n} z_i + q(\prod_{i=1}^{n} z_i)^{-1} : (\mathbb{C}^*)^n \to \mathbb{C}.$$

Kontsevich's HMS conjecture [28] predicts an equivalence[7]

$$D^b(\mathbb{P}^n) \cong FS((\mathbb{C}^*)^n, W)$$

between the derived category of $\mathbb{P}^n$ and the Fukaya-Seidel category [41] of Lefschetz fibration W. We denote a Lefschetz thimble of W to be Δ^n which is diffeomorphic to a n-dimensional disk.

The mirror of $K_{\mathbb{P}^n}$ [24] is the hypersurface

$$\check{X} = \{(x, y) \in (\mathbb{C}^*)^n \times \mathbb{C}^2 \mid y_1 y_2 + W(x) = z\},$$

where z is a regular value of W. In [42], Seidel introduced the suspension of Lefschetz fibrations and interpreted $\check{X}$ as the double suspension of a regular fiber of W. Under the double suspension, $\partial(\Delta^n)$ becomes a Lagrangian sphere L ($\cong \mathbb{S}^{n+1}$) in $\check{X}$. Then one obtains the mirror analog of (1)

$$HF^*_{\check{X}}(L, L) \cong HF^*_{(\mathbb{C}^*)^n}(\Delta^n, \Delta^n) \oplus HF^{n+1-*}_{(\mathbb{C}^*)^n}(\Delta^n, \Delta^n)^*,$$

where $HF^*_{(\mathbb{C}^*)^n}(\Delta^n, \Delta^n) \triangleq H^*(\Delta^n, \mathbb{Z})$.

3.2. **Cyclic A_∞-algebras in Lagrangian Floer theory.** We recall definitions of cyclic A_∞-algebras over a field $\mathbb{K}$ and their existences on Lagrangian Floer cohomologies which are needed for the completion of a proof of Theorem 3.1.

Definition 3.5. ([18]) An A_∞-algebra is a $\mathbb{Z}$-graded $\mathbb{K}$-vector space

$$A = \bigoplus_{p \in \mathbb{Z}} A^p$$

endowed with graded maps

$$m_n : A^{\otimes n} \to A, n \geq 0$$

of degree $2 - n$ such that for any $k \geq 0$, we have

$$\sum_{k_1+k_2=k+1} \sum_i (-1)^{deg(x_1)+\cdots+deg(x_{i-1})+i-1} m_{k_1}(x_1, \cdots, m_{k_2}(x_i, \cdots, x_{i+k_2-1}), \cdots, x_k)) = 0.$$

As we do not require $m_1^2 = 0$, it is sometimes called curved A_∞-algebra [27]. Following [20], we call an A_∞-algebra strict if $m_0 = 0$, in which case we have $m_1^2 = 0$. To reflect the Calabi-Yau n-algebra structure, we introduce the cyclic condition on A_∞-algebras.

Definition 3.6. ([18]) A finite dimensional A_∞-algebra (A, m_k) is called n-cyclic, if there exists a homogenous bilinear map

$$Q : A \otimes A \to \mathbb{K}[-n]$$

such that

- $Q(x, y) = (-1)^{(degx+1)(degy+1)+1} Q(y, x)$,
- $Q(m_k(x_1, ..., x_k), x_0) = (-1)^* Q(m_k(x_0, ..., x_{k-1}), x_k)$,

where $* = (deg(x_0) + 1)(deg(x_1) + \cdots + deg(x_k) + k)$.

A typical example of strict n-cyclic A_∞-algebra is the extension group of sheaves on compact Calabi-Yau n-folds [39], [30], [50]. The mirror analog in Lagrangian Floer theory is due to Fukaya [18] and Fukaya, Oh, Ohta and Ono [20].

We take a relatively spin compact Lagrangian submanifold L in a compact symplectic manifold X. The universal Novikov ring is

$$\Lambda_{0,nov} = \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} e^{n_i} \mid a_i \in \mathbb{Q}, \lambda_i \in \mathbb{R}_{\geq 0}, n_i \in \mathbb{Z} \text{ and } \lim_{i \to \infty} \lambda_i = \infty \right\},$$

with maximal ideal $\Lambda^+_{0,nov}$ which consists of elements such that $\lambda_i \in \mathbb{R}_{>0}$. If L has zero Maslov index and X is Calabi-Yau, $H^*(L; \Lambda^+_{0,nov}) = H^*(L; \mathbb{Q}) \otimes_{\mathbb{Q}} \Lambda^+_{0,nov}$ will have a $\mathbb{Z}$-graded cyclic A_∞-algebra structure, i.e.

[7] See Katzarkov, Kontsevich and Pantev [26] for a summary and Auroux, Katzarkov and Orlov [3] for some partial results.

Theorem 3.7. *(Fukaya* [18]*, Fukaya-Oh-Ohta-Ono* [20]*)*
Let L be a compact relatively spin Lagrangian submanifold of zero Maslov index inside a Calabi-Yau n-fold X [8]*. Then $H^*(L;\Lambda^+_{0,nov})$ has a n-cyclic A_∞-algebra structure with respect to the Poincaré pairing, which is well-defined up to isomorphisms.*

Finally, by combining Theorem 3.7 and Lemma 3.2, we finish the proof of Theorem 3.1.

4. Appendix on the B-model story$-DT_4$ theory

We recall some basic facts in Donaldson-Thomas theory on Calabi-Yau 4-folds. The main references are Borisov-Joyce's article [6] and the authors's preprints [9, 10, 11, 12].

We start with a compact Calabi-Yau 4-fold $(X,\mathcal{O}_X(1))$ $(Hol(X)=SU(4))$ with a Ricci-flat Kähler metric g [52], a Kähler form ω, a holomorphic four-form Ω, and a topological bundle with a Hermitian metric (E,h). We define

$$*_4=(\Omega\lrcorner)\circ * : \Omega^{0,2}(X,EndE)\rightarrow\Omega^{0,2}(X,EndE),$$

with $*_4^2=1$ and it splits the corresponding harmonic subspace into (anti-)self-dual parts.

The DT_4 equations are defined to be

$$\left\{\begin{array}{l} F^{0,2}_+=0 \\ F\wedge\omega^3=0, \end{array}\right. \tag{2}$$

where the first equation is $F^{0,2}+*_4F^{0,2}=0$ and we assume $c_1(E)=0$ for simplicity in the moment map equation $F\wedge\omega^3=0$.

We denote $\mathcal{M}^{DT_4}(X,g,[\omega],c,h)$ or simply $\mathcal{M}^{DT_4}_c$ to be the space of gauge equivalence classes of solutions to the DT_4 equations on E (with $ch(E)=c$).

We take $\mathcal{M}^{bdl}_c$ to be the moduli space of slope-stable holomorphic bundles with fixed Chern character c. By Donaldson-Uhlenbeck-Yau's theorem [51], we can identify it with the moduli space of gauge equivalence classes of solutions to the holomorphic HYM equations

$$\left\{\begin{array}{l} F^{0,2}=0 \\ F\wedge\omega^3=0. \end{array}\right. \tag{3}$$

By Lemma 4.1 [10], if $ch_2(E)\in H^{2,2}(X,\mathbb{C})$, then $F^{0,2}_+=0\Rightarrow F^{0,2}=0$. In particular, if $\mathcal{M}^{bdl}_c\neq\emptyset$, then $\mathcal{M}^{DT_4}_c\cong\mathcal{M}^{bdl}_c$ as sets. The comparison of analytic structures is given by

Theorem 4.1. *(Theorem 1.1* [10]*) We assume $\mathcal{M}^{bdl}_c\neq\emptyset$ and fix $d_A\in\mathcal{M}^{DT_4}_c$, then*
(1) there exists a Kuranishi map $\tilde{\tilde{\kappa}}$ of $\mathcal{M}^{bdl}_c$ at $\overline{\partial}_A$ (the (0,1) part of d_A) such that $\tilde{\tilde{\kappa}}_+$ is a Kuranishi map of $\mathcal{M}^{DT_4}_c$ at d_A, where

$$\tilde{\tilde{\kappa}}_+=\pi_+(\tilde{\tilde{\kappa}}) : H^{0,1}(X,EndE)\xrightarrow{\tilde{\tilde{\kappa}}}H^{0,2}(X,EndE)\xrightarrow{\pi_+}H^{0,2}_+(X,EndE)$$

and π_+ is projection to the self-dual forms;
(2) the closed imbedding between analytic spaces possibly with non-reduced structures $\mathcal{M}^{bdl}_c\hookrightarrow\mathcal{M}^{DT_4}_c$ is also a homeomorphism between topological spaces.

Remark 4.2. By Proposition 10.10 [10], the map $\tilde{\tilde{\kappa}}$ satisfies $Q_{Serre}(\tilde{\tilde{\kappa}},\tilde{\tilde{\kappa}})\geq 0$, where Q_{Serre} is the Serre duality pairing on $H^{0,2}(X,EndE)$.

To define Donaldson type invariants using $\mathcal{M}^{DT_4}_c$, we need to give it a good compactification (such that it carries a deformation invariant fundamental class). For this purpose, we introduce $\mathcal{M}_c(X,\mathcal{O}_X(1))$ or simply $\mathcal{M}_c$ to be the Gieseker moduli space of $\mathcal{O}_X(1)$-stable sheaves on X with given Chern character c. Motivated by Theorem 4.1, we make the following definition.

Definition 4.3. ([10]) We call a C^∞-scheme, $\overline{\mathcal{M}}^{DT_4}_c$ generalized DT_4 moduli space if there exists a homeomorphism

$$\mathcal{M}_c\rightarrow\overline{\mathcal{M}}^{DT_4}_c$$

such that at each closed point of $\mathcal{M}_c$, say $\mathcal{F}$, $\overline{\mathcal{M}}^{DT_4}_c$ is locally isomorphic to $\kappa_+^{-1}(0)$, where

$$\kappa_+=\pi_+\circ\kappa : Ext^1(\mathcal{F},\mathcal{F})\rightarrow Ext^2_+(\mathcal{F},\mathcal{F}),$$

κ is a Kuranishi map at $\mathcal{F}$ and $Ext^2_+(\mathcal{F},\mathcal{F})$ is a half dimensional real subspace of $Ext^2(\mathcal{F},\mathcal{F})$ on which the Serre duality quadratic form is real and positive definite.

[8] It is compact or convex at infinity with $c_1(X)=0$.

Remark 4.4.
1. The existence of generalized DT_4 moduli spaces is proved by Borisov-Joyce [6]. The authors proved their existence as real analytic spaces in certain cases and defined the corresponding virtual fundamental classes [9],[10].
2. For fixed data $(X, \mathcal{O}_X(1), c)$, the generalized DT_4 moduli space may not be unique. However, they all carry the same virtual fundamental classes.

The proof of Borisov-Joyce's gluing result is divided into two parts. Firstly, they used good local models of $\mathcal{M}_c$, i.e. local 'Darboux charts' in the sense of Brav, Bussi and Joyce [7]. Then they choosed the half dimensional real subspace $Ext^2_+(\mathcal{F},\mathcal{F})$ appropriately and used partition of unity and homotopical algebra to glue κ_+. We state an analytic version of the local 'Darboux charts' and give a proof using gauge theory.

Theorem 4.5. *(Brav-Bussi-Joyce* [7] *Corollary 5.20, see also Theorem 10.7* [10]*)*
Let $\mathcal{M}_c$ be a Gieseker moduli space of stable sheaves on a compact CY_4. For any closed point $\mathcal{F} \in \mathcal{M}_c$, there exists an analytic neighborhood $U_{\mathcal{F}} \subseteq \mathcal{M}_c$, a holomorphic map near the origin

$$\kappa : Ext^1(\mathcal{F},\mathcal{F}) \to Ext^2(\mathcal{F},\mathcal{F})$$

such that $Q_{Serre}(\kappa,\kappa) = 0$ and $\kappa^{-1}(0) \cong U_{\mathcal{F}}$ as complex analytic spaces possibly with non-reduced structures, where Q_{Serre} is the Serre duality pairing on $Ext^2(\mathcal{F},\mathcal{F})$.

Proof. (Proof of Theorem 10.7 [10]) We use Seidel-Thomas twists [25],[43] transfer the problem to a problem on moduli spaces of holomorphic bundles and then notice that

$$\int Tr(F^{0,2} \wedge F^{0,2}) \wedge \Omega = -8\pi^2 \int ch_2(E) \wedge \Omega = 0,$$

as E is holomorphic. □

The above theorem has an application to the unobstructedness of Gieseker moduli spaces.

Corollary 4.6. *(Corollary 10.9* [10]*) If for any closed point $\mathcal{F} \in \mathcal{M}_c$, $dimExt^2(\mathcal{F},\mathcal{F}) \leq 1$, then $\mathcal{M}_c$ is smooth, i.e. all Kuranishi maps are zero.*

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